The Deformation of Foamed Elastic Materials

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(With an Appendix by T. D. PENDLE)

1. INTRODUCTION

Notwithstanding the widespread use of foamed elastic materials, little attention appears to have been paid to their mechanics of deformation apart from the work of Conant and Wohler[1] and Talalay.[2] In particular, there seems to be no quantitative theory to account for the marked dependence of elastic properties on density and type of strain. In the present paper a theoretical treatment is presented on the basis of a model consisting of a large number of thin threads joined at their ends to form a three-dimensional network. Experimental measurements of the load-deformation relations for natural rubber foams prepared from latex are described also, and compared with the predictions of the theoretical treatment.

The load-deformation relations obtained for one rubber foam of relatively low density are shown in Figure 1; they are typical of those obtained on a variety of foamed elastic materials. In extension the relation is found to be substantially linear and a value of Young's modulus characteristic of the foam may be calculated. In Section 2 below the dependence of Young's modulus of the foam on the density is derived for the proposed model structure, and in section 3 measured values for a wide range of densities are compared with the predicted values.

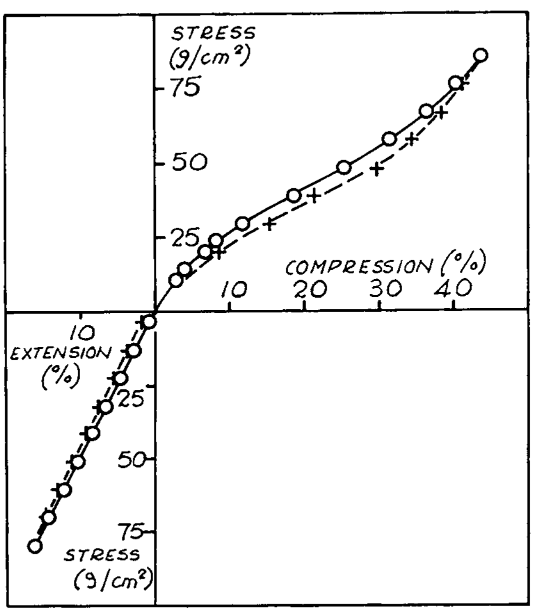

Fig. 1. Load-deformation relations in simple extension and compression for a natural rubber foam having a volume fraction of rubber of 0.125.

In compression the load-deformation relation is seen to be markedly nonlinear, resembling that for a typical collapsing process such as the buckling of a thin strut.[1] The proposed structure may be envisaged to undergo compression by buckling of the thin threads, and a corresponding treatment of the compression of a network is described in Section 4. Experimentally-determined load-deformation relations in compression are described in Section 5 and compared with the predictions of the theory.

The load-deformations were found to be somewhat irreversible as is seen in Figure 1. Although the separation between the load-increasing and load-decreasing curves is not large, it is considerably greater than that found in similar solid rubber vulcanizates. For consistency, only those measurements taken as the load increased have been considered below.

2. SMALL EXTENSIONS OF A NETWORK OF THIN THREADS

General Calculation

The model structure considered consists of a large number of thin threads of unstrained length l_0 and cross-sectional area A, connected at their ends by substantially undeformable volumes (Fig. 2). The presence of these dead volumes causes the strains in the threads to be greater than the average strain for the whole element consisting of thread and dead volume. If the diameter of the dead

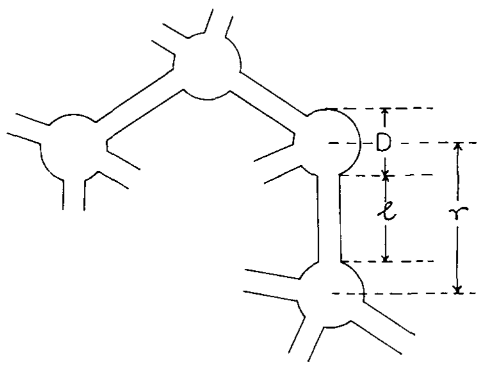

Fig. 2. Sketch of proposed model structure.

volume is D, and r_0 is the distance between the dead volume centers for a single element,

$$r_0 = l_0 + D$$

When strained, r_0 becomes r and

$$r = l + D$$

where l is the strained length of the thread.

For small strains it is assumed that no buckling occurs and that the threads are in simple extension or compression. The strain energy in the thread is therefore given by

$$w = {}^1/_2\, l_0\, YA\ (1 + D/l_0)^2\ [(r/r_0) - 1]^2 \quad (1)$$

where Y is Young's modulus for the material.

Assuming that the centers of the dead volumes move affinely with the bulk,

$$r^2 = r_0^2\ (\lambda_1^2 \sin^2\phi \cos^2\theta + \lambda_2^2 \sin^2\phi \sin^2\theta + \lambda_3^2 \cos^2\phi) \quad (2)$$

where λ_1, λ_2, and λ_3 are the bulk extension ratios, and θ and ϕ are polar coordinates.

For small strains, on putting $r = r_0 + \Delta r$, $\lambda_1 = 1 + e_1$, etc., we have

$$\Delta(r^2) = e_1 \frac{\partial(r^2)}{\partial\lambda_1} + e_2 \frac{\partial(r^2)}{\partial\lambda_2} + e_3 \frac{\partial(r^2)}{\partial\lambda_3}$$

Hence, from eq. (2),

$$2r\ (\Delta r/r_0^2) = 2\ (e_1 \sin^2\phi \cos^2\theta + e_2 \sin^2\phi \sin^2\theta + e_3 \cos^2\phi)$$

when λ_1, λ_2, and λ_3 are nearly unity. On substituting for $\Delta r/r_0$, eq. (1) takes the form

$$w = {}^1/_2\, l_0\, YA\ (1 + D/l_0)^2\ (r_0/r)^2\ [e_1 \sin^2\phi \cos^2\theta + e_2 \sin^2\phi \sin^2\theta + e_3 \cos^2\phi]^2 \quad (3)$$

As only small strains are considered, terms of order e^3 and above may be neglected. The factor $(r_0/r)^2$ is of order $(1 + e)$, whereas the last factor on the right-hand side of eq. (3) is of order e^2. The term $(r_0/r)^2$ can therefore be approximated to unity with the required accuracy.

The total strain energy is given by

$$W = \frac{2N}{\pi} \iint_0^{\pi/2} w \sin\phi \cdot d\theta \cdot d\phi$$

where N is the number of threads per unit volume. Substituting for w from eq. (3), and integrating,

$$W = N\, l_0\, YA\ (1 + D/l_0)^2\ [3(e_1^2 + e_2^2 + e_3^2) + 2(e_1e_2 + e_2e_3 + e_3e_1)]/30 \quad (4)$$

Equation (4) describes the behavior of the model structure under all types of deformation provided the strains are small. In particular, Poisson's ratio has the value $^1/_4$ and Young's modulus is given by

$$Y_F/Y = NA\ l_0(1 + D/l_0)^2/6 \quad (5)$$

In order to obtain a relation between Y_F and the volume fraction ν_r of material in the foam, a more detailed model structure must be assumed. However, in the limiting case when ν_r is small, it seems highly probable that D, the diameter of the dead volumes, will become negligible in comparison with l_0 whatever the detailed structure may be. Hence, when ν_r is small, $1 + D/l_0 \rightarrow 1$, $NAl_0 \rightarrow \nu_r$ and from eq. (5), $Y_F/Y = \nu_r/6$.

Particular Model

It is assumed that n threads, each of length l_0 and cross section A, enter each dead volume, and that each of these may be approximated by a sphere of surface area nA.

Thus

$$4\pi(D/2)^2 = nA$$

i.e.

$$D = (nA/\pi)^{1/2}$$

A representative volume of the foam may be taken as a sphere of radius $r_0/2$ concentric with a dead volume. The volume of this sphere is given by

$$V = {}^4/_3\ [(l_0/2) + (D/2)]^3$$

and the volume of material contained in it is

$$V_r = {}^1/_2\, nAl_0 + {}^4/_3\, \pi\, (D/2)^3$$

The volume fraction of material in the foam is thus

$$\nu_r = V_r/V = (3\beta^2 + \beta^3)/(1 + \beta)^3 \quad (6)$$

where

$$\beta = (nA/\pi\, l_0^2)^{1/2} = D/l_0$$

The factor NAl_0 in eq. (5) is the volume fraction of material in the foam which is in the form of threads, and is given by

$$NAl_0 = 3\beta^2/(1 + \beta)^3$$

The factor $(1 + D/l_0)^2$ is equal to $(1 + \beta)^2$. Substituting in eq. (5), we obtain

$$Y_F/Y = \beta^2/2\,(1 + \beta) \qquad (7)$$

Equations (6) and (7) thus determine Y_F/Y as a function of ν_r, since both are given as explicit functions of β. It is interesting to note that the quantities n, A, and l_0 do not appear explicitly in the final relations. Moreover, if a more restrictive model is assumed in which the dead volume is cubical in form, having six threads emerging from it, and the corresponding representative volume of the structure is taken to be a cube of side r_0 centered on the dead volume, the resulting relations for ν_r and Y_F/Y are found to be identical with those given in eqs. (6) and (7). It appears, therefore, that the form of these relations is not critically dependent on the detailed structure.

3. EXPERIMENTAL MEASUREMENTS AT SMALL DEFORMATIONS

Simple Extension

Samples of vulcanized natural rubber foam of varying densities were prepared by Mr. T. D. Pendle of these Laboratories in the form of sheets about 2 cm. thick. The mix formulation used and the method of preparation are given in the Appendix. The density of each foam was calculated from measurements of the dimensions and weight of a test-piece cut from each sheet, consisting of a prism about 20 cm. in length, 3 cm. in width, and 2 cm. in thickness. The test-pieces were then suspended from fixed upper clamps while tensile loads were applied by means of weights added to light lower clamps. The loads were increased by regular amounts, each load being maintained for about one minute before the corresponding extension was measured over the central region of the test-piece by means of a travelling microscope. The load-extension relations were found to be substantially linear over the range 0 to 10% extension and from the measured slopes values of Young's modulus were calculated for each foam. The values so obtained are given in Table I, together with the values of the volume fraction ν_r of rubber in the foams, calculated from the measured densities.

A sheet of solid rubber was obtained from the latex compound used in preparing the rubber foams by slowly drying the latex in a flat dish. It was then vulcanized by heating under similar conditions to those used in preparing the foams. In order to examine whether the foams and the solid rubber were vulcanized to a similar extent, measurements were made of the equilibrium swelling of test-pieces in benzene, the measured values of the equilibrium

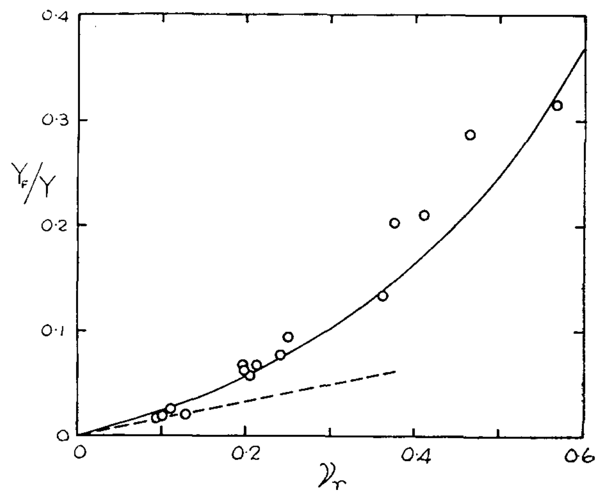

Fig. 3. Experimental relation between Young's modulus Y_F of the rubber foam, relative to that of the solid rubber Y, and the volume fraction ν_r of rubber in the foam. Full curve: Theoretical relation given by eqs. (6) and (7). Broken curve: Limiting form of theoretical relation for low foam densities.

linear swelling ratio λ_s being given in Table I. It is clear that an equivalent degree of vulcanization was achieved for the rubber foams and the solid rubber, and that little variation occurred between different samples of foam. The solid rubber vulcanizate, and the value of Young's modulus determined experimentally for it, namely 26.4 kg./cm.^{2}, may therefore be considered representative of the vulcanized rubber of which the foams are composed.

In Figure 3 the experimentally-determined values of the ratio Y_F/Y of Young's modulus of the rubber foam to that of the solid rubber are plotted against the values of the volume fraction ν_r of rubber in the foam. The theoretical relation given in eqs. (6) and (7) is represented by the full curve of Figure 3, and it is seen to describe the experimental results with considerable success.

It seems likely that the linear dimensions of the dead volumes in the proposed model structure will not be so large as the distances over which they effectively preclude any deformation. A slight modification of the theoretical treatment in which such an inequality was introduced was found to give improved agreement with the experimental results. However, in view of the somewhat arbitrary nature of the assumptions made and the satisfactory agreement obtained with the simple treatment, this extension of the theory is not considered further.

Poisson's Ratio

Measurements were made by means of a travelling microscope of the lateral contractions over the

TABLE I

Measurements in Simple Extension

Test-piece	Volume fraction of rubber, ν_r	Linear swelling ratio in benzene, λ_s	Young's modulus Y_F (kg./cm.^{2})	Y_F/Y	Poisson's ratio, σ
Solid rubber	1	1.560	26.4	—	—
Foam 1	0.093	1.583	0.425	0.0161	0.35
2	0.101	1.628	0.479	0.0182	0.43
3	0.110	1.586	0.644	0.0254	—
4	0.128	1.582	0.505	0.0192	0.32
5	0.196	1.562	1.79	0.068	0.35
6	0.197	1.595	1.67	0.063	0.33
7	0.205	1.569	1.51	0.057	0.33
8	0.223	1.589	1.77	0.067	0.29
9	0.240	1.564	2.03	0.077	0.34
10	0.249	1.588	2.50	0.095	0.32
11	0.362	1.522	3.55	0.135	0.35
12	0.401	1.570	5.36	0.204	0.34
13	0.422	1.589	5.56	0.211	—
14	0.464	1.578	7.58	0.288	0.28
15	0.568	1.580	8.32	0.315	0.31
				Mean value for σ:	*0.33*

central region of the test-piece when small extensions were imposed, of the order of 10%. Values of Poisson's ratio calculated from the measurements are given in Table I. No systematic trend with increasing density of the foam appears to exist, although considerable scatter is evident and is ascribed to the large errors involved in measuring small changes in dimensions by the present method. The average value obtained for σ was 0.33, compared with the theoretically predicted value of 0.25. The differences between load-deformation relations calculated using the two values would be insignificant.

4. COMPRESSION OF A NETWORK OF THIN THREADS

The model structure considered is similar to that used in Section 2 for the calculation of the behavior under small strains in so far as the elements themselves are concerned. However, in the present case a further simplification is necessary. The number of threads connected by each dead volume is assumed to be 6, and they are assumed to be directed perpendicular to each other, with one pair parallel to the direction of compression. Thus the compressive stress is taken by one-third of the threads, the other two-thirds being inert, and neither extended nor compressed. This assumption is consistent with experiment in as much as little alteration in the dimensions perpendicular to the compression is observed.

The deformation of the bulk is denoted by the extension ratio α ($\alpha < 1$), related to the extension ratio λ in the threads by

$$\alpha = (l + D)/(l_0 + D) = (\lambda + \rho)/(1 + \rho) \quad (8)$$

where $\lambda = l/l_0$, and $\rho = D/l_0$. The dead volumes are assumed to be cubes of side D, such that the cross-sectional area A of the threads is given by

$$A = D^2 \quad (9)$$

A representative volume is taken as a cube of side $l_0 + D$ concentric with one dead volume and with its edges parallel to the threads. The total volume enclosed is $(l_0 + D)^3$, including a volume of material given by $(D^3 + 3Al_0)$. Hence

$$\begin{aligned} \nu_r &= (D^3 + 3Al_0)/(l_0 + D)^3 \\ &= (\rho^3 + 3\rho^2)/(1 + \rho)^3 \end{aligned} \quad (10)$$

on substituting for A from eq. (9).

The deformation is attributed to buckling of the threads. The force F_b on each thread is therefore governed by the bending moments developed and takes the form

$$F_b = YAK^2 \cdot l_0^x \cdot f(\lambda)$$

where AK^2 is the moment of inertia of the thread cross section and $f(\lambda)$ is an unknown function of λ. It is clear on dimensional grounds that x is -2. The average stress across the face of the representative cubical volume is therefore given by

$$t = F_b/(l_0 + D)^2 = YAK^2 \cdot f(\lambda)/l_0^2\,(l_0 + D)^2 \quad (11)$$

If the shape of cross section of the threads remains similar for all the foams, then

$$AK^2 = mA^2$$

where m is a numerical constant. From eq. (9), therefore,

$$AK^2 = mD^4$$

Substituting for AK^2 in eq. (11) and absorbing the constant m in $f(\lambda)$, we obtain

$$t/Y = D^4 \cdot f(\lambda)/l_0^2 (l_0 + D)^2 = \rho^4 \cdot f(\lambda)/(1 + \rho)^2 \quad (12)$$

Equations (8), (10), and (12) give the form of the load-deformation relation in terms of an unknown function $f(\lambda)$. This latter should, however, be independent of Y and ν_r, and so in principle should be obtainable from a single compression curve at a particular value of ν_r.

When ν_r is small, ρ is small and ν_r is approximately given by $3\rho^2$. Also $\alpha \rightarrow \lambda$, so that t/Y at a given overall compression ratio should vary as ν_r^2.

Contribution to Deformation from Simple Compression

In addition to the deformation considered above, arising from buckling of the threads comprising the network, some contribution to the total deformation might be expected due to simple compression. Under small strain conditions the two components may be assumed additive. The corresponding bulk extension ratio α' will therefore be given by

$$1 - \alpha' = 1 - \alpha + t/Y_F \quad (13)$$

where α is the bulk extension ratio associated with buckling of the threads and Y_F is Young's modulus of the foam.

5. EXPERIMENTAL MEASUREMENTS IN COMPRESSION

Determination of $f(\lambda)$

From corresponding measurements of the compressive stress t and the ratio α' of the compressed height to the initial height of a test-piece of the foamed material, values of α may be calculated by means of eq. (13), using the value of Y_F found experimentally from measurements in simple extension. Corresponding values of λ may then be obtained from eq. (8).

From eq. (12) we have

$$f(\lambda) = (1 + \rho)^2 \, t/\rho^4 Y$$

Hence corresponding values of $f(\lambda)$ and λ may be determined from experimental measurements of t and α' on any one foam.

Measurements were made of the load-deformation relations in compression for test-pieces of several of the foams described in Section 3. The test-pieces consisted of prisms about 4 cm. in height and of square cross section having sides of length about 5 cm. The compressive loads were applied by means of weights, the corresponding deformations being measured with the aid of a travelling microscope. For the foams of lowest density, the term t/Y_F in eq. (13) was relatively small. The values calculated for α, and hence λ, may thus be considered more accurate for these materials. Corresponding values of $f(\lambda)$ and λ were therefore calculated as described above from the load-deformation relations obtained for six of the lightest foams. The six relations between $f(\lambda)$ and λ obtained in this way were found to be closely similar. In Figure 4 a composite curve representing the best fit to the experimentally-obtained relations is shown, the separate relations being omitted for clarity.

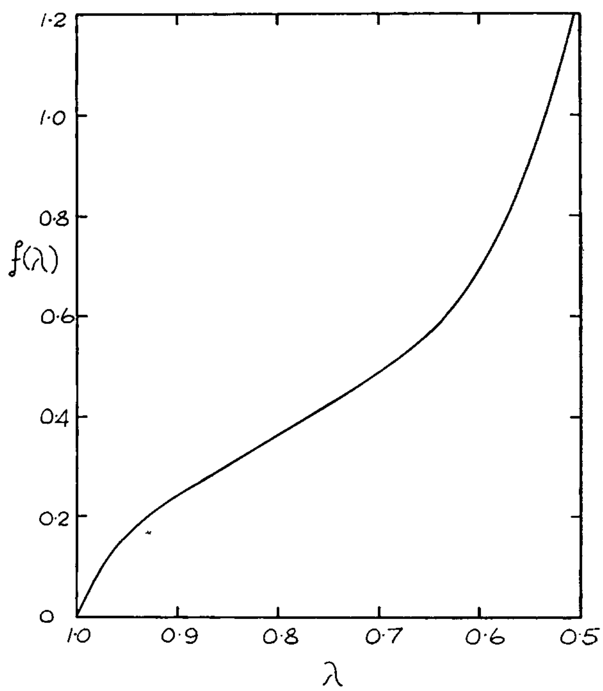

Fig. 4. Experimental relation between $f(\lambda)$ and λ, obtained from the load-deflection relations in compression for six of the lightest foams.

The relation shown in Figure 4 is of the general form which might be expected for a buckling process. However, for a simple system such as a thin strut in compression, the decrease in slope observed as λ decreases from unity would be expected to be much more pronounced, amounting to an abrupt reduction to zero at the axis $\lambda = 1$. The gradual decrease in slope shown in Figure 4 as λ decreases from unity presumably reflects the distribution of effective thread dimensions in the foams examined.

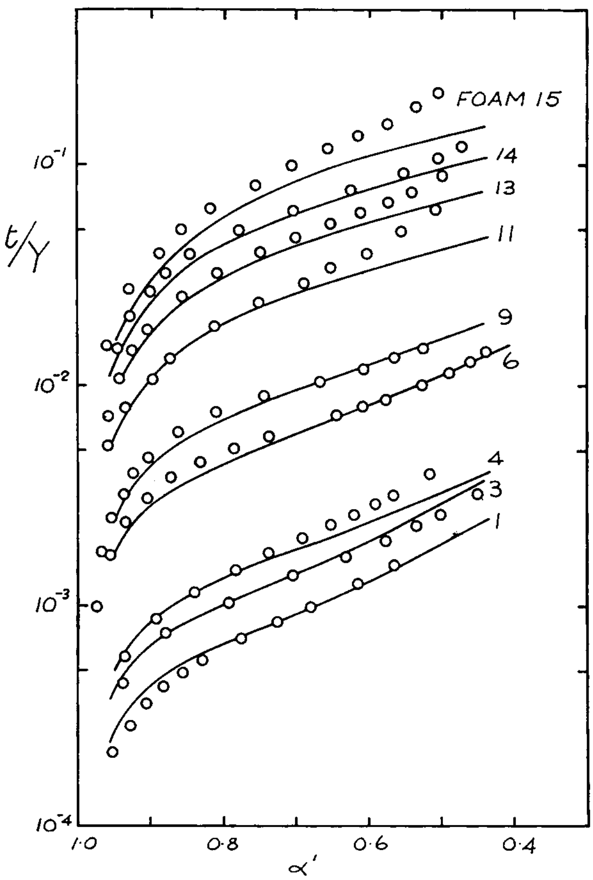

Fig. 5. Load-deflection relations in compression. The full curves are calculated from theory, using the values of $f(\lambda)$ given in Fig. 4.

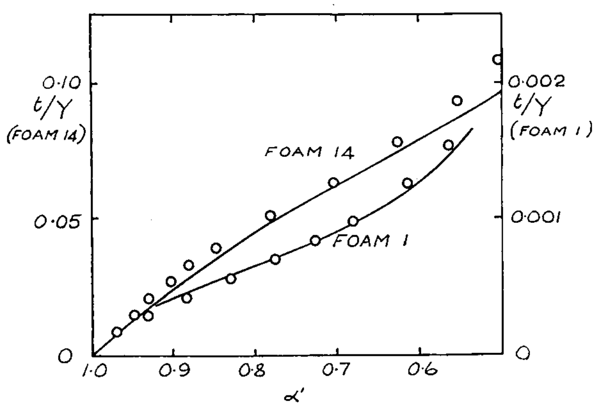

Fig. 6. Load-deflection relations in compression for Foam 14 (ν_r = 0.464) and Foam 1 (ν_r = 0.093). The full curves are calculated from theory using the values of f(λ) given in Fig. 4.

However, the good agreement found for foams of varied density indicates that the distribution is characteristic of the material and method of preparation employed. The relation obtained for $f(\lambda)$ may therefore be considered representative of latex foam rubber.

A more regular structure, such as that which appears to exist in polyurethane foams, might be expected to yield a relation for $f(\lambda)$ more closely resembling the ideal buckling form. Although the imperfect elastic behavior of polyurethane materials precludes a satisfactory experimental determination of $f(\lambda)$, the load-deformation relations obtained in compression appear to be in accord with such a relation since they exhibit the nonlinearity characteristic of collapsing processes to a considerably greater extent than the latex foam materials.

Comparison of Experimental Measurements with Theory

From the experimentally-obtained composite curve for $f(\lambda)$ as a function of λ given in Figure 4, it is possible to calculate the compressive stress t at any value of λ by means of eq. (12) and the corresponding bulk extension ratio α' by means of eqs. (8) and (13). The load-deflection relations calculated in this way for the foams examined are represented by the full curves of Figure 5. The applied loads are plotted on a logarithmic scale in view of the wide range of values employed.

The experimental measurements of load and deformation are represented by open circles in Figure 5; considering the wide range in hardness, about 100 to 1, covered by the foams, the agreement with the calculated relations is very satisfactory. There are also variations in the shape of the load-deformation curve for foams of different densities which the theory attempts to account for. In Figure 6 the experimental measurements for foams 1 and 14 are plotted on linear scales, together with the theoretical relations. The theory predicts the qualitative change in form satisfactorily.

Some departures from the theory may be observed in Figure 5, particularly for the dense materials at large compressions. Under these conditions the simple compression component of the deformation [t/Y_F in eq. (13)] becomes large, and at high strains the assumption that this component is proportional to t is probably seriously in error.

6. DISCUSSION OF RESULTS

The theoretical model used is a very idealized representation of an actual foam rubber, which is far from homogeneous, the threads of rubber and the interstices being of a wide range of sizes and shapes. Such variations cannot easily be taken into account in a theory, and no attempt has been made to do so. In view of this, the agreement obtained with theory for the Young's modulus of materials of different densities (Fig. 3) is very satisfactory, particularly as no arbitrary constants are involved, and suggests that the basic concepts of the structure and mode of deformation are correct.

In the case of compression, the detailed structure and the inhomogeneities of the foam probably affect the shape of the stress-strain curve markedly. This leads to the presence of an arbitrary function $f(\lambda)$ which will be a suitable average of these effects. However, the theory gives unequivocally the variation of hardness with density, and in view of the wide range in hardness covered, the agreement may be considered good. This supports the premise that the deformation is primarily due to buckling of the elementary threads.

This work forms part of a program of research undertaken by the Board of the British Rubber Producers' Research Association. The authors are indebted to Mr. T. D. Pendle for preparation of the samples of latex foam.

APPENDIX
PREPARATION OF LATEX FOAM SAMPLES

T. D. Pendle

The samples of foam were all prepared by the sodium silicofluoride method from a 60%, centrifuge-concentrated, ammoniated natural rubber latex. The ammonia content was reduced to 0.2% by the addition of formaldehyde solution, and the following ingredients were added per 167 g. of latex, i.e., per 100 g. rubber:

10% solution of potassium ricinoleate	7 g.
5% solution of cetyl trimethylammonium bromide	3 g.
20% solution of potassium chloride	2.5 g.

The latex was whisked into a foam by means of a mechanical beater, the density of the products being varied by varying the amount of foaming. A 50% dispersion of vulcanizing ingredients and a 25% dispersion of sodium silicofluoride were then added to give the following quantities per 100 g. of rubber:

Sulfur	2.5 g.
Zinc oxide	3 g.
Zinc diethyl dithiocarbamate	1 g.
Zinc 2-mercaptobenzothiazole	0.3 g.
sym-Di-β-naphthyl-*p*-phenylene diamine	0.5 g.
Sodium silicofluoride	1 to 1.5 g.

The foam was then poured into a mold and allowed to set. Vulcanization was effected by heating for 30 minutes at a temperature of 100°C. in steam. The vulcanized foam was finally washed thoroughly in cold water and dried in an air oven at 60°C.

In order to produce foams of density greater than 0.3 g./cm.^{3}, the above formulation had to be modified slightly. The amount of foaming agent (potassium ricinoleate) was drastically reduced, and up to 1.5% of a thickening agent (ammonium polymethacrylate) was added to increase the viscosity of the foam.

References

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2. Talalay, J. A., *Ind. Eng. Chem.*, **46,** 1530 (1954).

Synopsis

A theoretical treatment is given which predicts the behavior of a foamed elastic material on the basis of a model consisting of a network of thin threads. Two cases are considered: (*1*) small strains, and (*2*) finite compressions, when the major part of the deformation of the threads is attributed to buckling. The behavior is given in terms of Young's modulus of the matrix and the density of the foam. Measurements of the load-deformation relations for small tensile strains and finite compressions are described for natural rubber foams prepared from latex. A wide range of density is covered (0.09–0.57), giving a variation of compression hardness of about 100:1. Satisfactory agreement with theory is found for both the cases considered, indicating that the basic concepts of the structure and mode of deformation are correct.

Résumé

Un traitement théorique est donné en vue de prévoir le comportement d'un matériau spongieux élastique sur la base d'un modèle consistant en un réseau de fils fins. Deux cas sont considérés: (*1*) faibles tensions et (*2*) compressions déterminées, lorsque la partie principale de la déformation des fils est due au plissement. Le comportement est exprimé sous la forme du module de Young de la matrice et de la densité de la masse. Des mesures des relations charge-déformation sous de faibles forces de tension et des compressions limitées sont décrites pour des mousses de caoutchouc préparés au départ de latex. Un grand domaine de densité est couvert (0.09–0.57), permettant une variation de force de compression d'environ 100:1. Un accord satisfaisant avec la théorie a été trouvé pour les deux cas considérés, ce qui démontre que les concepts de base pour la structure et les modes de déformation sont corrects.

Zusammenfassung

Es wird eine theoretische Untersuchung durchgeführt, die unter Zugrungelegung eines Modells, das aus einem Netzwerk dünner Fäden besteht, die Voraussage des Verhaltens eines elastischen Schaumstoffes gestattet. Zwei Fälle werden betrachtet: (*1*) kleine Verformung und (*2*) endliche Zusammendrückung, bei welcher der Hauptteil der Deformation der Fäden einer Verbiegung zugeschrieben wird. Das Verhalten wird als Funktion des Young-Moduls der Matrix und der Dichte des Schaumes beschrieben. Messungen der Beziehung zwischen Belastung und Deformation bei kleinen Zugspannungen und endlicher Zusammendrückung werden für Naturkautschuk-Schaumstoffe wiedergegeben, die aus Latex dargestellt wurden. Es wird ein weiter Dichtebereich untersucht (0.09–0.57), entsprechend einer Variierung der Kompressionshärte etwa im Verhältnis 100:1. Für beide in Betracht gezogene Fälle wird befriedigende Übereinstimmung mit der Theorie gefunden, was dafür spricht, dass die Grundannahmen bezüglich der Struktur und der Art und Weise der Verformung korrekt sind.

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